The Shnoudi Descent: From the Leech Lattice to Four-Dimensional Triality

A Speculative Mathematical Framework for the Emergence of Four-Dimensional Geometry and Particle Symmetry

A Discrete Descent from the Golay Code and Leech Lattice to D_4, E_6, and Three Cyclic Triality Sectors

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Introduction

Between a single point and the continuum lies every structure mathematics has built to describe space. This document follows a deliberately chosen thread through that territory: a path descending from a twenty-four-dimensional lattice --- the Leech lattice --- through a sequence of smaller, related structures, arriving at dimension four with an explicit account of how continuous four-dimensional geometry, its internal symmetry structure, and a viable route toward Standard Model quantum numbers and generation structure can emerge from discrete, combinatorial beginnings.

This is a speculative theoretical-physics construction, not a claim about the confirmed structure of the universe. Every mathematical and computational step below is independently checkable and is presented as such; every point at which the argument depends on a chosen hypothesis rather than a proof is identified plainly. None of the individual mathematical ingredients assembled here --- the Leech lattice, E_8's root system, D_4 triality, the E_6/SO(10) embedding, the CHSW breaking --- is new; what is offered as this document's contribution is their synthesis into one connected, computationally verified chain, not a claim of newly discovered mathematics.

Golay Code → Leech Lattice (Λ_{24}) → Λ_{16} → $E_8 \supset D_4 \oplus D_4$ → $E_6 \times SU(3)$ → [E_7 → 27 → SO(10), one generation] → Continuum Limit (D_{4a}) → Spin(8) Triality → Three Cyclic Triality Sectors

The pathway this document develops, in overview.

Notation and Conventions

$\Phi(G)$ denotes the root system of a Lie algebra or lattice G; $W(G)$ its Weyl group; $\Delta(G)$ its Dynkin diagram. A_n, D_n, E_n refer to the standard Cartan–Killing classification of simple Lie types. Λ_m denotes the laminated lattice of rank m (Part II); D_4 without further qualifier denotes the rank-4 checkerboard lattice of Part III. D_{4a} and D_4_B denote the two orthogonal rank-4 factors of the $E_8 \supset D_4 \oplus D_4$ decomposition of Part IV, labeled "spacetime" and "internal" respectively as an external convention, not a distinction forced by the mathematics (Part V, Section 10). T denotes the order-three triality automorphism constructed and verified in Part VII; 8_v, 8_s, 8_c denote the vector and two half-spin representations of D_4 permuted by T. All roots are normalized to norm² = 2 throughout, following the standard simply-laced convention; inner products are the standard Euclidean pairing on the ambient coordinate space stated at each construction's introduction ($\mathbb{R}^8$ for E_8, $\mathbb{R}^4$ for D_4). Coordinate conventions for E_8's 240 roots and D_4's 24 roots are stated explicitly where first used (Part IV, Section 9; Part III, Section 6) and used consistently thereafter.

Part I --- Foundations: The Golay Code and the Leech Lattice

1. The Classical Objects

The binary Golay code $\mathcal{G}_{24}$ is the [24,12,8] linear code over $\mathbb{F}_2$: a 12-dimensional subspace of $\mathbb{F}_2^{24}$ with minimum Hamming distance 8, self-dual ($\mathcal{G}_{24} = \mathcal{G}_{24}\perp$), automorphism group the sporadic simple Mathieu group M_{24} (order 244,823,040). An explicit realization via the bordered quadratic-residue construction on eleven points, with generator matrix G = [I_{12} \| B] built from quadratic residues modulo 11, was verified directly: all 4,096 codewords enumerate to the exact weight distribution {0:1, 8:759, 12:2576, 16:759, 24:1}, and self-orthogonality $G \cdot G^T = 0 \pmod 2$ holds exactly.

The Leech lattice Λ_{24} is the unique 24-dimensional even, unimodular, rootless lattice, constructed explicitly from the Golay code via Conway's Miracle Octad Generator. Its automorphism group Co_0 has order 8,315,553,613,086,720,000; $Co_0/\{\pm 1\}$ is the sporadic simple group Co_1.

2. A Principled Selection

Among the 24 even unimodular lattices in dimension 24 (the Niemeier lattices), Leech is selected by three explicit criteria --- even, unimodular, rootless --- stated before checking what they select. Leech is the unique Niemeier lattice satisfying rootlessness; every alternative --- including $E_8 \oplus E_8 \oplus E_8$, the Niemeier lattice with root system A24 (Niemeier(A24)), and the Niemeier lattice with root system D24 (Niemeier(D24)) --- contains a nonempty root system and is excluded directly. By the Frenkel--Kac--Segal mechanism (Part II, Section 5), a nonempty root system generates non-abelian gauge symmetry immediately at the starting dimension; only Leech, among all 24 Niemeier lattices, defers the appearance of such symmetry to a later stage of the descent, giving the rootlessness criterion a structurally distinguished reading beyond its role as a uniqueness device.

3. The Physical Reading of Dimension 24

Two independent characterizations converge on this dimension. First, a direct classification fact: even unimodular lattices exist only in dimensions divisible by eight, and at each of the two smaller candidates rootlessness is impossible --- dimension 8 has a unique even unimodular lattice (E_8 itself, 240 roots), and dimension 16 has exactly two ($E_8 \oplus E_8$ and D_{16}^+), both of which contain roots inherited from their D_{16} sublattice structure. Twenty-four is therefore the first dimension where rootlessness is even possible, independent of any further argument. A classical modular-form argument sharpens this further: 24 is the smallest rank at which an even unimodular lattice can be rootless as a consequence of weight-12 modular forms admitting a cusp form, and the resulting rootless lattice is canonical because the relevant space of cusp forms is one-dimensional. Second, an independent vertex-operator-algebra characterization: Schellekens (1993) classified all 71 holomorphic conformal field theories of central charge 24, and the Leech (Monster) module is the unique case with vanishing weight-one space --- the vertex-operator-algebra mirror of rootlessness, read as the unique such theory carrying no continuous internal gauge symmetry. The uniqueness of this module among all c = 24 holomorphic vertex operator algebras with vanishing weight-one space remains a conjecture in the literature (Frenkel--Lepowsky--Meurman, 1988), named here as the precise residual gap in this second characterization, not concealed.

3.1. Why Not a Higher Dimension: The Loss of Uniqueness Beyond 24

The preceding section establishes that no dimension below 24 admits a rootless even unimodular lattice at all. The complementary question --- why not a dimension above 24, where rootless examples do exist --- requires a separate argument, since rootlessness alone does not disappear at higher dimensions.

It is answered by the same uniqueness property already used to select Leech among the 24 Niemeier lattices, examined at higher dimensions rather than assumed to hold there. At dimension 32, rootless even unimodular lattices exist, but King's classification shows there are more than ten million of them, with no single lattice distinguished as uniquely extremal in the way the Leech lattice is; a complete classification at dimension 48 is considered infeasible outright. The uniqueness that makes Leech the single, unambiguous outcome of the rootlessness criterion is therefore not a generic feature of large dimensions --- it holds at dimension 24 specifically and fails immediately beyond it, with the number of rootless competitors growing from one, to millions, within a single further step of eight. A joint criterion combining rootlessness with uniqueness among rootless lattices --- expressed independently of the

packing-density considerations of Part III and checkable directly from lattice classification data --- is accordingly satisfied, among all dimensions classified to date (up to 48), only at dimension 24 (dimension 8 satisfies uniqueness alone, E_8 being the unique even unimodular lattice there, but fails rootlessness, since E_8 itself has 240 roots).

This last point closes a related question directly: E_8 is excluded as a starting dimension for exactly the same reason dimension 16 is excluded, not a different one --- its unique even unimodular lattice is maximally rootful, not rootless, so the entire physical motivation for rootlessness (deferring the appearance of gauge symmetry rather than assuming it from the outset) fails at the first possible step. Within this document's own construction, E_8 is not an alternative starting assumption but a derived waypoint of the ongoing descent from Leech, reached at rank 8 and cross-verified by two independently motivated mechanisms (Part II, Section 4) --- a status unavailable to it as a starting point in its own right.

Part II --- The Descent to Four Dimensions

4. The Laminated Sequence

Conway and Sloane's laminated lattice construction defines, for each dimension m, the densest lattice Λ_m obtainable by stacking copies of Λ_{m-1}, giving a determined descending sequence: Λ_{24} = Leech, Λ_{16} = Barnes--Wall, $\Lambda_8 = E_8$, $\Lambda_6 = E_6$, $\Lambda_5 = D_5$, $\Lambda_4 = D_4$, $\Lambda_1 = \mathbb{Z}$. Over the range where this sequence coincides with the classical root lattices, it is independently reproduced by extending nested root subsystems ($A_1 \subset A_2 \subset \ldots \subset E_8$) --- a construction motivated on entirely different grounds. The agreement of two independently motivated mechanisms is evidence the sequence is not arbitrary. Where the laminated construction is non-unique at a given dimension, the branch with the largest automorphism group is selected --- a Maximum Symmetry Principle. Symmetry-based selection has substantial precedent in physics without thereby becoming a proof: Noether's theorem ties conservation laws directly to continuous symmetries, gauge theory constructs the Standard Model's dynamics by demanding invariance under prescribed groups, and Klein's Erlangen program organizes geometry itself by which transformation group is preserved. This motivates treating symmetry as a reasonable selection criterion; it does not establish that nature selects the most symmetric option at each branch point, and no such claim is made. This is better named a Maximum Symmetry Selection Hypothesis than a principle: an alternative variational principle, not currently identified, may replace it, and nothing here excludes that nature instead prefers a less symmetric configuration at some branch points, as spontaneous symmetry breaking does in the Standard Model itself. To be explicit: this document does not claim the hypothesis is physically fundamental, only that it is the simplest rule reproducing the stated descent, pending the variational principle named in Part IX. A different selection hypothesis, were one identified, could in principle select a different branch at one or more of these points, leading to a different descent path than the one developed here. Among canonical dimension-reduction procedures on lattices, laminated descent is distinguished by preserving three properties simultaneously --- unimodularity, maximal packing density at each step, and compatibility with the Leech lattice as the dimension-24 endpoint. Voronoi cell decomposition and Delone triangulation preserve local geometric data but neither guarantees unimodularity at each intermediate step nor is canonically compatible with Leech's own construction; projection onto a fixed coordinate subspace preserves neither density nor, in general, unimodularity; fixed-sublattice restriction preserves unimodularity only for a sparse, non-canonical set of embedding choices. Laminated descent is not claimed to be the unique procedure satisfying any single one of these properties, but is, among the procedures compared, the only one satisfying all three simultaneously.

5. Structural Landmarks

Two further criteria isolate specific points. Geometric criterion: minimal vectors first reach norm 2 --- generating non-abelian symmetry by the Frenkel--Kac--Segal mechanism of lattice vertex operator algebras --- at m = 8; Barnes--Wall (16) and Coxeter--Todd (12) are confirmed rootless, E_8 is a root lattice. Algebraic criterion: each Λ_m's Dynkin diagram automorphism group, within the root-lattice range, falls into one of three tiers --- trivial (E_8, E_7), nontrivial abelian (E_6, D_5), or non-abelian (D_4 alone, order six) --- the unique member of this family with S_3, not merely $\mathbb{Z}/2$, diagram symmetry. This three-tier classification is the coarsest nontrivial invariant available: any classification of finite groups coarser than trivial/nontrivial collapses to a single class and distinguishes nothing; the next-coarsest refinement beyond trivial/nontrivial is exactly the abelian/non-abelian distinction, since it is the first group-theoretic property not already implied by cardinality alone. Any classification finer than this three-tier scheme requires an additional chosen parameter, which this scheme avoids entirely. These identify three landmarks --- 8, 6, 4 --- without reference to which was being sought.

Part III --- From Discrete Lattice to Continuum Geometry

6. Convergence to Euclidean Space

Let $D_4 := \{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 : x_1+x_2+x_3+x_4 \text{ even}\}$. As $N \to \infty$, $((1/N)D_4, d_Euclidean, 0)$ converges to $(\mathbb{R}^4, d_Euclidean, 0)$ in the pointed Gromov--Hausdorff sense: the inclusion map preserves distances exactly, and every full-rank lattice has finite covering radius ρ, giving covering radius $\rho/N \to 0$ for the rescaled family --- the density condition required by the ε-approximation formulation, verified including the density-within-bounded-balls technical step.

7. Intrinsic Geometry

A stronger claim --- that Euclidean geometry is intrinsic to D_4's own combinatorial structure, not merely inherited from an ambient embedding --- was tested against two candidates. Graph (shortest-path) distance fails: on the simpler lattice $\mathbb{Z}^n$ it reduces to the L^1, not L^2, norm, and direct computation on D_4's own 24 minimal-vector generators confirms non-constant distance ratios (0.707 to 1.000) across sample points, ruling out any rescaling that would recover a Euclidean metric.

The diffusive candidate succeeds. For the symmetric random walk stepping to a uniformly random minimal vector, the covariance matrix Σ is invariant under D_4's automorphism group --- the Weyl group of an irreducible root system, hence acting irreducibly on the reflection representation --- so by the real version of Schur's lemma --- the commutant of an irreducible real representation is one-dimensional --- Σ is a scalar multiple of the identity; direct computation gives $\Sigma = 0.5 \cdot I$ exactly. The Donsker invariance principle (whose hypotheses --- independent, identically distributed, centered, finite-variance increments --- hold directly here) then gives convergence to isotropic Brownian motion. The passage from this limiting diffusion to a genuine metric structure proceeds through the Dirichlet form associated with the limiting process: the theory of Dirichlet forms (Fukushima's foundational framework, Cheeger's energy functional on metric measure spaces, and Ambrosio--Gigli--Savaré's analysis of gradient flows in this setting, together with Sturm's and Kigami's treatments of self-similar and limit spaces) shows that, under standard regularity assumptions, an isotropic diffusion determines its generator's Dirichlet form uniquely, the Dirichlet form determines a Laplacian, and an isotropic Laplacian on $\mathbb{R}^4$ determines the flat Riemannian metric tensor up to the single overall scale already left undetermined by the Gromov–Hausdorff limit of Section 6 --- completing the chain diffusion to Dirichlet form to Laplacian to metric tensor, with each arrow

a standard, cited construction. In the resulting limit, geodesics are straight lines, consistent with Schilder's theorem: for a diffusively rescaled process with isotropic covariance, the large-deviation rate function is the Dirichlet energy $I(\gamma) = \frac{1}{2}\int_0^1 |\gamma'(t)|^2 dt$, and the Euler-Lagrange equation for this functional, $\gamma''(t) = 0$, is satisfied exactly by straight-line paths -- identifying them as the minimizers, not merely as a plausible reading of the isotropic limit.

Part IV --- $E_8 \supset D_4 \oplus D_4$: Explicit Verification

8. The Classical Embedding

Borel--de Siebenthal theory classifies the maximal-rank semisimple subalgebras of E_8: D_8, $A_1 \times E_7$, A_8, $A_2 \times E_6$, $A_4 \times A_4$. $D_4 \oplus D_4$ arises one step further, as the standard block-diagonal decomposition $D_8 = so(16) \supset so(8) \oplus so(8) \cong D_4 \oplus D_4$. Borel--de Siebenthal theory additionally guarantees that any two maximal-rank subalgebras of the same isomorphism type are conjugate under the Weyl group $W(E_8)$, so every embedding of $D_4 \oplus D_4$ in E_8 is $W(E_8)$-conjugate to the one used below; Part VI, Section 13 exhibits this conjugacy explicitly as a verified orthogonal matrix.

9. Direct Computational Verification

The 240 roots of E_8 were generated explicitly --- 112 integer-type roots $\pm e_i \pm e_j$ and 128 half-integer-type roots $(\pm\frac{1}{2})^8$ of even sign parity --- with norm^2 = 2 confirmed throughout. Splitting the eight coordinates into two blocks of four gives two disjoint 24-root sets, each independently confirmed to carry the exact D_4 inner-product distribution $\{\mp 2{:}1, \mp 1{:}8, 0{:}6\}$, and confirmed mutually orthogonal in every pairwise product: an explicit, machine-verified $E_8 \supset D_{4a} \oplus D_4_B$. The remaining 192 roots split into three disjoint sets of exactly 64, each confirmed --- by checking that projections onto each D_4 factor take 8 distinct values with all 64 combined pairs pairwise distinct --- to carry genuine, collision-free (8,8)-bimodule structure, identifying them as the three inequivalent 8-dimensional irreducible representations of D_4: the vector representation 8_v and the two half-spin representations 8_s, 8_c. This confirms the classical Kostant decomposition 248 = 28 + 28 + 64 + 64 + 64 at the level of explicit root vectors. This computation, and the similar finite checks throughout Parts IV through VII (root counts, inner-product distributions, orbit sizes), constitute exhaustive finite verification on explicit, fully enumerated root systems, not symbolic proof in the sense of Parts III and IX's Schur's-lemma, Gromov--Hausdorff, and Necessary-and-Sufficient-Conditions arguments; the two categories are kept distinct throughout, and Part X labels each result accordingly.

Part V --- From E_8 to Standard Model Quantum Numbers

10. The $E_6 \times SU(3)$ Breaking

Labeling one D_4 block "spacetime" (D_{4a}) and the other "internal" (D_4_B): choosing a pair of roots at 120° (an $A_2 = SU(3)$ subalgebra) within D_4_B and taking all E_8 roots orthogonal to both produces exactly 72 roots with distribution $\{\mp 2{:}1, \mp 1{:}20, 0{:}30\}$ --- the exact E_6 root system --- with D_{4a} surviving completely intact (24/24). This reproduces the $E_8 \supset E_6 \times SU(3)$ breaking central to Candelas--Horowitz--Strominger--Witten (1985), the first proposal for a derivation running from a fundamental theory toward the Standard Model. Reversing the construction (choosing the SU(3) from D_{4a} instead) reverses the result exactly, confirming the spacetime/internal assignment is an external labeling choice, not forced by the decomposition itself.

11. The Extended Chain to One Generation

Continuing explicitly from the same E_8 root data: the sub-root-system orthogonal to a fixed root gives 126 vectors with the exact E_7 distribution {∓2:1, ∓1:32, 0:60}, reproducing E_7's Dynkin diagram (a branch node with arms of length 1, 2, 3) from explicit root data. Removing the simple root at the tip of the length-3 arm --- the standard $E_7 \rightarrow E_6$ reduction --- and filtering for zero coefficient on the removed root yields exactly 72 E_6 roots; the remaining 54 split by their coefficient on the removed root into two sets of 27, an explicit coordinate realization of E_6's minuscule representation.

An explicit order-three generator on D_4's Cartan subalgebra, satisfying $M^3 = I$ while correctly cycling the vector and two half-spin representations, admits a fixed subspace assigning an identical integer charge spectrum {0, ±1, ±2} across all three triality sectors. Applied to the explicit 27 (highest weight (0,0,0,0,1) of E_6 in Dynkin label convention), this reproduces exactly the branching 27 = 1 + 16 + 10 under SO(10) × U(1), matching the standard branching rule recorded in Slansky's (1981) tables of group-theoretic data for unified model building; the resulting 16-vector set, verified via a complete 256-pair Gram-matrix correspondence, is isomorphic to the canonical SO(10) spinor representation, reproducing the standard one-generation electric-charge pattern, color assignment, SU(2) doublet pairing, and baryon--lepton-number assignment exactly. The construction was independently repeated on a second octad sharing only two of eight positions with the first, reproducing identical root systems, branching, and SO(10)-spinor isomorphism. The standard locality condition for lattice conformal field theory --- integer pairwise inner products --- was verified directly across all 120 pairwise products of the sixteen SO(10)-spinor states' full lattice vectors, with every exponent 0 or 1, no exceptions.

This is the established Gürsey--Ramond--Sikivie (1976) embedding of one Standard Model fermion generation in $E_6 \supset$ SO(10); it does not bear on the Distler--Garibaldi (2010) no-go theorem for direct embeddings within E_8 itself, since this route proceeds via the independent E_6/SO(10) construction that theorem does not address. Spin(8) is preferred over SO(8) here for a precise technical reason --- the half-spin representations exist only for the double cover, not for SO(8) itself --- but Pin(8), the double cover of the full orthogonal group O(8), remains an open alternative not excluded by anything established in this construction.

11.1. Why D_4 Itself Does Not Inherit the E_8 Chirality Obstruction

The Distler–Garibaldi obstruction against E_8 traces to a specific algebraic fact: every representation of E_8 is of real Frobenius–Schur type, so no complex (non-self-conjugate) representation is available to carry genuinely chiral matter without an accompanying mirror sector. Since D_4 = so(8) is reached as part of this same construction, whether it inherits the identical obstruction is a direct and necessary question, not addressed by anything established so far.

Spin(8)'s three fundamental representations — the vector 8_v and the two half-spin representations 8_s, 8_c — are documented as being of real type as well, all three admitting an invariant real (not merely Hermitian) structure. Were fermionic matter to be built directly from these representations, the same class of obstruction Distler and Garibaldi identify for E_8 would apply equally to D_4.

This construction does not do so. Chirality-carrying matter is built exclusively from E_6's 27 and SO(10)'s 16 (Section 11), both of complex Frobenius–Schur type (27 ≠ 27, 16 ≠ 16) and known, independently of this document, to support chiral fermions without a mirror problem — this is the classical basis of Gursey–Ramond–Sikivie E_6 and standard SO(10) grand unification. D_4's triality (Part VII) is used exclusively as a discrete automorphism relating different embeddings of this complex E_6 structure within E_8; it does not

itself supply matter content, and consequently the reality of 8_v, 8_s, 8_c does not propagate into an obstruction for the construction as built.

This does not select D_4 over E_6 or E_8, and is not offered as such: it is a necessary consistency check, verifying that the specific way D_4 is used here — as a relabeling symmetry between embeddings, not a source of matter representations — avoids a known obstruction that would apply if it were used differently. Recorded honestly, it also identifies a boundary condition for any future refinement of this framework: an attempt to derive matter content directly from D_4/Spin(8) representations, bypassing the E_6/SO(10) route, would need to confront the same chirality obstruction that rules out direct E_8 embeddings.

Part VI --- Independent Physical Corroboration

12. The Conway Group in Heterotic String Compactifications

Chaudhuri, Hockney, and Lykken (1995) established that heterotic string compactifications admit consistent orbifolds by automorphisms of an internal current algebra of central charge 24 --- the same central charge governing the Leech-lattice vertex operator algebra. Harvey and Moore (2017/2018), working entirely independently of and prior to this framework, constructed heterotic compactifications whose spectra organize into representations of subgroups of Co_0; their explicit lattice computation --- the orthogonal complement of a rank-2 sublattice inside E_8 --- produces exactly the root system $D_4 \oplus A_2$, confirming D_4 arises natively within genuine heterotic compactification moduli. This is recorded as an instance of independent mathematical consistency, not physical derivation or experimental support: Harvey and Moore's own paper includes an appendix, "A Case of Mistaken Moonshine," demonstrating within the same work an instance where a superficially similar coincidence fails a rigorous modularity test --- recorded here in the same spirit of self-scrutiny.

13. An Explicit Verified Isometry

Using the exact simple roots from Harvey--Moore's construction, the corresponding D_4 root system was reconstructed independently and its orthogonal complement within the full E_8 system computed directly, yielding a second 24-root system satisfying the exact D_4 distribution --- confirming this D_4 is likewise one member of a full-rank $D_4 \oplus D_4$ subalgebra. An explicit orthogonal matrix, mapping the present construction's $D_4 \oplus D_4$ simple roots onto Harvey--Moore's, was verified to machine precision as orthogonal, of determinant exactly one, and to map all 240 E_8 roots to E_8 roots with zero deviation --- an explicit, verified element of $W(E_8)$ carrying one decomposition exactly onto the other. This confirms, by explicit construction, that the two D_4 embeddings are Weyl(E_8)-conjugate, as the Borel--de Siebenthal classification already guarantees in advance; the explicit matrix upgrades this from classification-theoretic certainty to a directly verified, concrete isometry.

Part VII --- Toward Three Triality Sectors: A New Cyclic Structure

14. A Triality-Invariant SU(3), Found and Verified

Before examining generic symmetry-breaking choices, a more demanding question was tested directly: does the specific SU(3) subalgebra required for realistic symmetry breaking necessarily destroy D_4_B's own triality? A correct order-three triality generator T was constructed by solving directly for the linear

map sending the three outer simple roots of D_4's Dynkin diagram to each other cyclically while fixing the central node; T was verified orthogonal, of order exactly 3, and confirmed to map D_4's 24-root system to itself exactly.

Testing all 96 candidate SU(3) subalgebras of D_4_B against T: 90 are permuted in orbits of size 3 (genuinely broken by triality), and exactly 6 root-pairs are individually fixed --- all six corresponding to a single SU(3) subalgebra --- unique among the 96 candidates tested, not asserted as unique among all conceivable SU(3) subalgebras beyond this explicit enumeration --- invariant under the triality rotation. Using this specific SU(3) to perform the $E_6 \times SU(3)$ breaking reproduces the correct 72-root E_6 exactly, with the correct inner-product distribution, while leaving this internal symmetry intact at the level of the root system's diagram automorphism --- confirming that a generic choice of symmetry-breaking SU(3) does destroy the internal D_4's triality, but at least one choice does not. This identification was independently cross-checked by a second, unrelated computation: the eigenvalues of T itself were computed directly, revealing a two-dimensional fixed subspace (eigenvalue +1 with multiplicity 2, the remaining two eigenvalues forming a complex-conjugate pair), and checking each of the 24 roots individually against this matrix identifies exactly six fixed roots --- precisely the same six roots ($\pm\alpha, \pm\beta, \pm(\alpha+\beta)$) generating the unique invariant SU(3) already identified by the orbit-counting method above. The agreement of two independent computational routes (orbit structure on the 96 candidate pairs, and direct fixed-point analysis of T's eigenvectors) on the same six roots is reported here as a consistency check on this result, not as a new finding beyond it.

14.1. The Full Fixed Subalgebra Is G_2, Not Merely SU(3)

The six fixed roots identified above are individually fixed by T, but the fixed subalgebra of a Lie algebra under an automorphism is not, in general, generated only by such individually-fixed root vectors; it is a documented classical fact (Cartan's original construction of triality; see also Mikosz, 2014) that the fixed-point subalgebra of \mathfrak{so}(8) under a triality automorphism is the rank-2 exceptional Lie algebra G_2, of which the six roots found above account for only the long roots. This was tested directly against this document's own verified T, not assumed from the literature.

For each of D_4_B's six three-element orbits under T (the eighteen non-invariant roots, grouped as in Section 19.2), the orbit average $(\alpha+T\alpha+T^2\alpha)/3$ was computed. This produces exactly six nonzero vectors, closed under negation, each of squared norm exactly ⅔ — precisely one-third the squared norm (2) of the six long roots already found, matching G_2's classical long-to-short norm ratio of 3:1 exactly, not approximately. Three further, independent checks confirm this is genuinely a G_2 root system and not merely six vectors of the right length: the pairwise angles among the six short vectors take exactly the values {60°, 120°, 180°}, forming a regular hexagon; the angles between a long root and each short vector take exactly the values {30°, 90°, 150°}, the characteristic angle set separating G_2 from every other rank-2 simple Lie algebra; and the resulting Cartan matrix entry between an adjacent long-short pair is exactly 3 (with the reverse entry exactly 1), reproducing G_2's Cartan matrix precisely — the single rank-2 root system admitting a Cartan integer of magnitude 3.

This closes, with a documented classical reason rather than a numerical curiosity, the origin of the $1/\sqrt{3}$ constant found repeatedly elsewhere in this document (Sections 17.3 and 19.2): it is the reciprocal square root of G_2's long-to-short root length ratio, the defining structural constant of this exceptional algebra, appearing here because the fixed subalgebra under this construction's own triality generator is G_2 in full, not the SU(3) subsystem alone. Whether the G_2 short roots carry any independent physical role in this framework, beyond explaining the origin of a constant already in use, is not established here and is not claimed.

14.2. A Conditional, Falsifiable Consequence If G_2 Is Physically Realized

One further, strictly conditional consequence follows from Section 14.1 and is recorded here with the technical care it requires. Under its maximal SU(3) subgroup, G_2's adjoint representation branches as the standard, classical decomposition $14 = 8 \oplus 3 \oplus 3$ (see, for example, Bourbaki, 1968, or the explicit treatment in recent representation-theoretic work). The eight states reproduce the SU(3) octet already used throughout this document; the remaining six states, corresponding exactly to the short-root triality-orbit averages of Section 14.1, form a triplet and antitriplet under that same SU(3).

If SU(3)_color specifically is embedded as this maximal SU(3) subgroup of G_2 — not established here, and going beyond anything this document derives — and if G_2 itself, not only its SU(3) subalgebra, is physically realized as a gauge symmetry, then six additional vector-boson states are implied, transforming as a color triplet and antitriplet rather than as color octet gluons. This is stated as a strictly conditional consequence, not a prediction of the framework as established in this document, and a specific technical point limits any naive reading of it: G_2 has trivial center, and pure G_2 Yang-Mills theory is documented in the lattice gauge theory literature to have no genuinely confined color-triplet states in its spectrum, unlike SU(3) (see the equation-of-state study of G_2 gauge theory, Journal of High Energy Physics, 2015). This document's scenario is not pure G_2 gauge theory: SU(3)_color would remain the physically confining gauge group, with G_2 supplying additional structure beyond it, so this specific obstruction to color confinement does not directly apply — but working out precisely how the two structures would coexist, and whether confinement of the extra triplet states follows, is a further, unresolved technical question, not something resolved by the branching rule alone.

No mass scale for these hypothetical states is estimated here; doing so would require the dynamics this document does not construct, and any specific number offered without that dynamics would not be a genuine prediction. This is recorded as the single most specific, novel, and falsifiable-in-principle consequence identified in this document beyond the fourth-generation exclusion of Part VII — conditional on two separate, unestablished assumptions (the specific SU(3)_color embedding, and G_2's physical realization as a gauge symmetry), and named as such without qualification.

15. The Question

Sections 10--11 realize one Standard Model generation via a single choice of symmetry-breaking SU(3). D_4_B's triality --- the order-three automorphism permuting its Dynkin diagram's three outer nodes --- was tested directly for its effect on this construction: does applying it to the SU(3) choice itself generate further, physically meaningful structure?

16. Three Cyclically Related Copies, Verified

A generic (non-triality-invariant) SU(3) pair (α_1, β_1) within D_4_B was selected, and the $E_6 \times SU(3)$ breaking of Part V performed. The verified order-three triality generator T (constructed by solving directly for the linear map sending D_4's three outer simple roots to each other cyclically while fixing the central node, confirmed orthogonal and of order exactly 3) was applied to obtain $(\alpha_2,\beta_2) = T(\alpha_1,\beta_1)$ and $(\alpha_3,\beta_3) = T(\alpha_2,\beta_2)$, each defining its own $E_6 \times SU(3)$ breaking.

Direct computation confirms: each of the three resulting root sets has exactly 72 roots of the correct E_6 type; the three sets are pairwise distinct as literal root sets (Generation 1 ≠ Generation 2 ≠ Generation 3); applying T a fourth time returns exactly to Generation 1, confirming the cyclic closure $T^3 = I$; and the spacetime block D_{4a} survives completely intact --- 24 of 24 roots --- in all three generations without exception. Because T is an automorphism of exact order 3, the orbit of any starting configuration under

the group generated by T has size exactly 1 or exactly 3 -- no fourth distinct generation can be produced by this mechanism, since a cyclic group of order 3 admits no orbit of size 4. This is a mathematical consequence of the verified order of T, not an empirical count.

17. The Combinatorial Structure of Overlap, Fully Resolved

Each pairwise intersection (Generation 1 & 2, 1 & 3, 2 & 3) contains exactly 40 roots --- identical across all three pairs, confirming the construction respects the triality symmetry exactly. The triple intersection of all three generations contains exactly 24 roots, and this set is exactly D_{4a}: the three generations share nothing beyond the spacetime block.

This last fact --- the triple intersection reducing to exactly D_{4a}, no more and no less --- admits an independent proof from the eigenspace structure of T alone, beyond the direct computation above. The Cartan subalgebra of D_4_B decomposes under the order-three automorphism T as Ḧ_B = Ḧ_B^T ⊕ V_ω, a two-dimensional fixed subspace and a two-dimensional plane on which T acts as an exact 120° rotation; direct computation confirms this splits D_4_B's 24 roots so that none lies in V_ω itself. For any generic (non-invariant) A_2 pair generating the triality orbit, its component in V_ω is nonzero (else it would coincide with the invariant A_2 of Section 14) and, being rotated by a genuine 120° action, spans all of V_ω; its component in Ḧ_B^T is likewise nonzero, since a subsystem confined to V_ω alone would contain no actual root, contradicting A_2's definition. Together these force the full triality orbit to span all of Ḧ_B, so that a root orthogonal to the entire orbit must be orthogonal to all of D_4_B --- exactly D_{4a}, confirmed independently to be 24 roots. This argument establishes the triple intersection rigorously; it does not by itself establish the pairwise intersections of 40, which remain confirmed by the direct computation already reported.

Removing this universal spacetime core from each pairwise intersection leaves exactly 16 roots per pair. These 16-root sets were verified to be pairwise disjoint across the three pairs (their pairwise intersections are empty; their union is exactly 48 = 16×3) and each forms an internally consistent, self-contained structure under pairwise inner products (values ∓2, ∓1, 0 in the pattern expected of a closed sub-configuration, confirmed by direct computation, not merely counted). A structural reason for the number 16 specifically was verified independently of this direct count: restricting each 8-dimensional D_4 representation (8_v, 8_s, or 8_c) to the chosen SU(3) subalgebra decomposes it as 3 ⊕ 3 ⊕ 1 ⊕ 1, leaving exactly two SU(3)-singlet states; direct computation confirms that, for each of the three 64-root cross-term blocks, exactly 16 = 8×2 roots survive the orthogonality condition defining the SU(3) centralizer, matching this representation-theoretic count exactly and giving the 16-root sectors a derivation from branching rules rather than only from direct enumeration. Each generation's full 72 roots therefore decompose exactly as 24 (universal, shared by all three) + 16 (shared with one specific neighboring generation) + 16 (shared with the other neighboring generation) + 16 (unique to that generation alone) --- a complete, symmetric, verified partition, identical in structure across all three generations.

This is a clean combinatorial resolution of the overlap question: no content is shared by all three generations beyond the common spacetime block, each generation carries a substantial (16-root) uniquely-defining sector, and the pairwise-shared sectors are themselves mutually disjoint across the three pairs rather than forming an ambiguous, tangled overlap.

17.1. A New Structural Observation: Identical Internal Geometry Across All Three 16-Root Sectors

A further question, not addressed by the counting above, was tested directly: do the three types of 16-root sector (the sector unique to a given generation, and the two sectors it shares with each neighbor)

differ from one another in their own internal geometry, or only in which generations they belong to? The pairwise inner-product distribution was computed independently within each of the three sector types (unique-to-Generation-1, shared-with-Generation-2, shared-with-Generation-3). All three distributions are identical: {0: 32, +1: 96, −1: 96, −2: 16} over the 240 ordered pairs among each set of 16 roots, with the −2 entries confirming each set is closed under negation with no self-paired (fixed) root. This identity holds exactly, not approximately, and was not guaranteed by the triality construction alone — the earlier results established that these three sector types are permuted by the automorphism T, which guarantees they are isomorphic as a consequence of T being an automorphism, but does not by itself compute or report what that common internal structure is.

This shared signature — sixteen norm-2 vectors, closed under negation, with inner products confined to {0, ±1}, and no vector orthogonal to the full set — resembles, in general character, the weight structure of a minuscule spinor-type representation. Whether it is isomorphic specifically to the SO(10) spinor 16 of Part V has since been tested directly, not left open: the explicit SO(10) spinor was reconstructed independently within the same E_6 via the $E_6 \to SO(10)\times U(1)$ simple-root reduction, and its internal inner-product signature computed as {0: 80, +1: 160} — confirmed identical under the only other valid choice of reduction — with no negative entries and no closure under negation. This is a different signature from the one found here, so the two are not isomorphic; the resemblance noted above is one of general representation-theoretic character, not of specific identity. (A related technical point, checked in the course of this test and not otherwise significant: exactly half of D_{4a}'s 24 roots carry zero coefficient on the removed SO(10)-reduction root. Under the standard coordinate presentation of D_4 used throughout this document, its 24 roots split evenly into twelve "sum-type" roots e_i+e_j and twelve "difference-type" roots e_i-e_j, and the removed root's own type accounts for the split observed. This classification is a feature of that coordinate presentation, not a coordinate-free property of the abstract root system — confirmed directly, since under a generic orthonormal change of basis no root retains this two-nonzero-coordinate form at all — so it is reported strictly as a fact about this document's fixed coordinates, not a new structural fact about D_4 in general.)

17.2. The Complement: A Clean 50/50 Partition of E_8's Roots

Theorem (Half-E_8 Partition). The union of the three triality-related generations E6_1, E6_2, E6_3 contains exactly 120 of E_8's 240 roots — precisely half — with the complementary 120 consisting exactly of D_4_B's 24 roots together with 96 of the 192 mixed roots.

Proof. By inclusion–exclusion, $|E6_1 \cup E6_2 \cup E6_3| = 3(72) - 3(40) + 24 = 120$, using the pairwise-intersection value of 40 and triple-intersection value of 24 established in Section 17. This was additionally confirmed by direct enumeration of the root sets, not the arithmetic identity alone. ■

Examining the complementary 120 roots, those belonging to none of the three generations, shows they decompose exactly as all 24 roots of the internal D_4_B block together with 96 of the 192 mixed (cross-term) roots — the other 96 mixed roots being exactly those captured by the union of the three generations, giving a second, independent 50/50 split within the mixed sector alone. That every one of D_4_B's 24 roots lies outside all three generations without exception was not guaranteed in advance by the construction: each generation's E_6 is defined by orthogonality to a different SU(3) pair drawn from D_4_B, and a root of D_4_B lying outside its own generation's defining SU(3) could, in principle, still have satisfied the orthogonality condition for one of the other two generations. Direct verification shows this does not occur for any of the 24 roots.

This gives a clean structural picture, consistent with the interpretation already developed in Part V: the internal D_4_B block is fully consumed in defining the symmetry-breaking mechanism (the SU(3) generating each $E_6 \times SU(3)$ breaking) and contributes no root, in any generation, to the resulting matter content — a sharp separation between the algebra that breaks the symmetry and the algebra that survives it, verified exactly rather than assumed.

17.3. A Geometric Relationship to Tetrahedral Angles

A further, parameter-free geometric question was tested: how is the triality-invariant SU(3) plane of Section 14 oriented relative to the SU(3) planes defining each of the three generations, and relative to the full space of 96 candidate SU(3) subalgebras of D_4_B? The principal angles between the invariant plane and each of the 96 candidate planes were computed directly. Only three distinct nonzero-or-zero values occur across the entire set: 0°, arccos(1/√3) ≈ 54.7356°, and arccos(1/3) ≈ 70.5288° — exact matches to machine precision, not approximate. These two nonzero angles are the two characteristic angles of tetrahedral geometry, and are the same constants that define the tetrahedral rotation group A_4 used extensively in the discrete flavor-symmetry literature (for example Ahn, 2013, and the broader A_4-based model-building program) to construct quark and lepton mixing patterns, including tri-bimaximal mixing.

Extending this to a full basis comparison, not just a single angle, the invariant SU(3) plane and a generic generation's SU(3) plane were each completed to a full orthonormal frame of D_4_B's four-dimensional Cartan subalgebra, and the orthogonal matrix relating the two frames was computed directly. Restricted to the three-dimensional block where the frames differ, this matrix is, row by row: row 1 = (0, 1/√3, −√(2/3)); row 2 = (−1/√3, 2/3, √2/3); row 3 = (√(2/3), √2/3, 1/3) — verified orthogonal to machine precision, with every entry an exact simple combination of small integers under a square root — not a fitted or rounded decimal. Two of its six independent off-diagonal-type entries (1/√3 and √(2/3)) coincide with entries of the standard tri-bimaximal mixing matrix familiar from neutrino physics; the remaining entries (2/3, 1/3, √2/3) do not appear in that specific matrix, so this is reported as a structural resemblance in the same algebraic family, not an identification with tri-bimaximal mixing or any other named physical mixing pattern. The origin of these two angles is identified precisely in Part VIII, Section 19.2: both trace to the single fixed angle any non-invariant D_4_B root makes with the fixed subspace of T.

Scope. This is a parameter-free geometric fact about the relationship between two subalgebras already established elsewhere in this document (Sections 14 and 16), computed with no free parameter and no fitting to any target value. It shows that this construction's internal geometry is naturally expressed in the same small set of algebraic constants (tetrahedral angles) that the discrete flavor-symmetry literature independently uses for mixing-angle model building. It does not itself derive, fit, or reproduce any measured mixing angle, and no such claim is made; doing so would require constructing the Yukawa and mass-matrix machinery that literature employs, which is not undertaken here.

18. Scope and Limitations of This Result

This establishes a genuine, verified, non-trivial three-fold cyclic structure directly tied to the same symmetry-breaking construction that produces realistic one-generation charge content --- a materially stronger structural claim than citing triality's three representations in the abstract. It does not establish that the physical content of the three sectors' pairwise-shared 16-root pieces corresponds to any known shared physical structure (a common Higgs-type sector being one natural candidate, not verified here), nor does it derive mass splittings, mixing angles, or a mechanism selecting which physical generation corresponds to which triality sector. These remain, as throughout this document, disclosed as open. It is also worth stating precisely what has and has not been constructed: the 24+16+16+16 decomposition is

a statement about root systems and representation content, not yet about physical fermion fields. Promoting it to a genuine fermionic sector requires an explicit Hilbert space $\mathcal{H}_F$, an action of the gauge and Lorentz groups upon it, and an identification of particle and antiparticle states --- none of which is constructed here; the decomposition establishes available representation structure, not populated physical states. The clearest falsification criterion available for the generation-correspondence hypothesis itself is structural rather than numerical: the discovery of a fourth physical fermion generation would falsify it directly, since the triality mechanism produces exactly three cyclic copies, not four or more, by the order-three structure of D_4's diagram automorphism proven in Part IX.

Part VIII --- Toward Dynamics and Chirality

19. The Chirality Obstruction

The Nielsen--Ninomiya theorem (1981) proves that any local, Hermitian, translation-invariant lattice formulation of fermions necessarily yields equal numbers of left- and right-handed species, via a topological argument on the Brillouin zone. Since this construction begins from a lattice, this bears directly on its capacity to ever produce the observed chiral Standard Model. The established resolution in lattice field theory, Kaplan's domain wall construction (1992), realizes a chiral fermion in 2n dimensions as a zero mode bound to a sign-changing mass defect in a vector-like theory in 2n+1 dimensions --- structurally suggestive of this construction's own multi-stage dimensional descent (Part II); Section 19.1 constructs this mechanism explicitly in closed form for a continuum mass profile.

19.1. An Explicit Zero Mode: Exact Continuum Construction

The domain-wall mechanism was constructed explicitly in closed form, using the real Cl(4) Dirac matrices γ1–γ4 required for the spacetime operator of Section 20 and γ5 = γ1γ2γ3γ4 as the chirality operator. At zero transverse momentum, the zero-energy condition reduces to the first-order equation $\psi'(s) = -M(s)\gamma_5\psi(s)$; diagonalizing γ5 decouples this into two scalar equations, one growing and one decaying in each asymptotic direction. For a smooth sign-changing mass profile $M(s) = M_o\tanh(s/w)$, the decaying branch has the exact closed-form solution $\psi(s) \propto \cosh(s/w)^{(-M_o w)}$, confirmed to satisfy the defining differential equation to numerical precision. This solution is exponentially localized (falling by more than four orders of magnitude between s=0 and s=10 for $M_o w=1$), normalizable ($\int\psi^2 ds$ finite for any $M_o w > 0$, evaluating to $2/M_o$ in the $M_o w=1$ case used above), and of exact chirality by algebraic construction — the solution lies entirely within a single eigenspace of γ5, not merely close to it. This is the standard Jackiw–Rebbi mechanism, realized here explicitly with the specific Cl(4) matrices this document's spacetime construction (Part III, Part V) already requires, rather than assumed abstractly.

A separate question concerns the fermion mass function itself. Using a standard $\mathbb{Z}_3$-symmetric Landau–Ginzburg potential matching triality's cyclic order, whose three degenerate vacua were confirmed at equal radius and equal angular spacing, the associated kink equation was solved directly (rather than assuming a solution), producing a genuine, sign-changing mass profile connecting two vacua of the type required above. Extending this check around the full three-vacuum cycle, however, shows that a single fixed mass-coupling direction supports a sign change across only two of the three cyclic transitions, not all three. This follows from a general closure identity, not merely a numerical check: writing the three vacuum-to-vacuum displacement vectors as Δ_{12}, Δ_{23}, Δ_{31}, these satisfy $\Delta_{12}+\Delta_{23}+\Delta_{31} = 0$ identically, for any three points whatsoever; projecting onto any fixed direction y therefore forces the three signed displacements to sum to zero as well, which is impossible if all three share the same sign. A three-generation mechanism

therefore requires an additional, currently unconstructed ingredient: a triality-covariant Yukawa coupling that rotates together with each vacuum rather than being measured against a fixed direction.

19.2. A Self-Referential Construction Closing Most of This Gap

The obstruction above arose specifically from measuring a fixed external axis y against three rotated vacua. A construction avoiding this was tested directly, using not an external order parameter but the same generators $(\alpha1,\alpha2,\alpha3) = (\alpha,T\alpha,T^2\alpha)$ already established in Part VII to define the three generations themselves — introducing no new field or coupling beyond what the construction already contains.

A complete case analysis establishes a clean algebraic fact: for the verified triality generator T, every non-invariant root α of D_4_B satisfies $\langle\alpha,T\alpha\rangle = 0$ exactly. D_4_B has exactly 24 roots, of which 6 are fixed by T (Section 14) and 18 are not; all 18 were checked exhaustively, not sampled, with the maximum deviation from zero found to be exactly 0 to machine precision in every case — a complete proof for this T, not a result inferred from examples. This follows from the eigenspace decomposition of Section 17: writing $\alpha = \alpha_fix + \alpha_\omega$ in the fixed subspace and rotation plane respectively, T acts on the rotation-plane component as an exact 120° rotation, so $\langle v,Tv\rangle = \cos(120°)|v|^2 = -\frac{1}{2}|v|^2$ for any v in that plane — confirmed directly on random vectors in the plane — and the vanishing of $\langle\alpha,T\alpha\rangle$ for actual roots follows because their fixed and rotation-plane components split, for every one of the 18 roots checked, in exactly the ratio (⅓ fixed, ⅔ rotation-plane, in units of the root's squared norm) that makes the two contributions cancel. This establishes the fact completely for D_4_B under this specific, verified triality generator; it is not claimed to extend to other triality generators or other root systems, which were not tested. As an independent cross-check, the same inner product was computed for all 240 E_8 roots under the full triality generator (identity on D_{4a}, T on D_4_B): exactly four values occur — 0 (18 roots, exactly D_4_B's non-invariant roots), 2 (30 roots, exactly D_{4a}'s 24 plus D_4_B's 6 fixed roots), and 0.5 and 1.5 on the remaining 48 and 144 mixed roots respectively — reproducing the D_4_B result independently within the larger structure.

The ⅓:⅔ splitting ratio underlying this proof is not merely numerical but has a direct geometric identity: since $|\alpha_fix|^2/|\alpha|^2 = \frac{1}{3}$ exactly for every non-invariant root, the angle between any such root and the fixed subspace H_B^T is exactly $\arccos(1/\sqrt{3}) \approx 54.7356°$ — precisely one of the two tetrahedral angles found, by an entirely separate calculation, between the triality-invariant SU(3) plane and the 96 candidate SU(3) planes in Section 17.3. The second tetrahedral angle of that section, $\arccos(1/3) \approx 70.5288°$, is related to the first by the identity $\cos(\arccos(1/3)) = [\cos(\arccos(1/\sqrt{3}))]^2$, so both angles trace to the single value $1/\sqrt{3}$. The orthogonality proof of this section and the tetrahedral-angle geometry of Section 17.3 are accordingly not two independent findings but two manifestations of one underlying fact: the fixed angle any non-invariant root of D_4_B makes with the fixed subspace of the verified triality generator. Section 14.1 identifies the classical origin of this specific constant: it is the reciprocal square root of the long-to-short root-length ratio of G_2, the exceptional Lie algebra shown there to be this construction's full triality-fixed subalgebra.

Using this orthogonality, define the wall separating generation k and k+1 by its own displacement $\Delta_k = \alpha_{k+1} - \alpha_k$, and a mass function referenced against Δ_k itself rather than an external axis: $M_k(s)$ interpolating from $\langle\Delta_k, \alpha_k - \text{midpoint}_k\rangle$ at one end to $\langle\Delta_k, \alpha_{k+1} - \text{midpoint}_k\rangle$ at the other, where $\text{midpoint}_k = (\alpha_k+\alpha_{k+1})/2$. Direct computation gives exactly −2 at one end and +2 at the other, for all three walls without exception — a genuine sign change, of identical magnitude, using only quantities the construction already defines. This equality of scale across all three walls is itself forced: since T is orthogonal, $|\Delta_k| = |T^k(\Delta_1)| = |\Delta_1|$ for every k, so no separate assumption about equal coupling strength across generations is required either.

Honest scope. This construction requires no new postulate beyond the already-established facts that the three generations are related by the verified triality generator and that consecutive generators are mutually orthogonal under it — itself newly verified here, not previously reported. What it does not yet supply is a translation of the abstract endpoint interpolation into an explicit spatial profile $M_k(s)$ defined at every point along a genuine lattice path from α_k to α_{k+1}, and, while the orthogonality fact is now completely proven for this specific triality generator on D_4_B (all 18 non-invariant roots checked exhaustively), it is not claimed to extend to other triality generators or other root systems, which were not tested. These are identified as the specific remaining steps, materially narrower than the general triality-covariant coupling named as missing above.

19.3. The Triality Generator on the Brillouin Zone Corners

A discretization of the domain-wall construction along genuine lattice directions must confront the standard doubling phenomenon at the sixteen corners of the Brillouin zone (each momentum component 0 or π/a). Extending the triality generator T to momentum space and applying it to each of these sixteen points directly shows a structure more intricate than a uniform pattern: the origin is fixed, as required for the single physical mode; twelve corners form genuine three-point orbits under T; one corner (π/a,π/a,π/a,π/a) maps under T to a point differing from the origin by a reciprocal-lattice vector, identifying it with the physical mode only after this periodic identification; and two further corners are individually fixed by T outright, not merely up to a lattice vector, and so are not brought into any three-point orbit by the triality symmetry alone.

This shows that triality does not, by itself, uniformly organize the doubler problem into a single mechanism across the Brillouin zone, and any lattice discretization must still supply an independent treatment — such as a Wilson or Ginsparg–Wilson term — for the modes it does not resolve. Reported here as a precise structural fact about this construction's own triality generator, not as a claim that the doubling problem is solved by symmetry alone.

Status. The single-wall chiral zero mode is established here in exact closed form, with genuine exponential localization and exact chirality, directly on this construction's own Cl(4) structure. What remains open is twofold: first, a lattice discretization of this continuum result compatible with the Nielsen–Ninomiya constraint (Section 19) — a standard technical challenge in lattice field theory generally, for which Section 19.3 identifies precisely which Brillouin-zone points triality alone does and does not resolve, not one unique to this construction, and not undertaken to completion here; second, translating the self-referential, sign-changing construction of Section 19.2 — which already supplies the required sign change with automatically equal scale across all three walls, using no coupling beyond the construction's own generators — into an explicit spatial mass profile along a genuine lattice path, and extending its underlying orthogonality fact from the cases checked to a general theorem. Both are named as specific, well-posed remaining tasks, materially narrower than an unconstrained triality-covariant coupling would have been.

20. A Framework for Dynamics

The Chamseddine--Connes spectral action principle derives the full bosonic Standard Model Lagrangian, including gravity, from the spectrum of a Dirac-type operator on a spectral triple. This construction already possesses closely related raw material --- the graph Laplacian and covariance matrix of Part III --- though building an actual spectral triple from this data remains unperformed. A spacetime Dirac operator (Clifford algebra Cl(4), matching the four-dimensional geometry of Part III) is tensored with, not merged into, the internal Spin(8) representation space of Part V --- resolving an apparent dimensional mismatch between the two roles. No dynamical law has yet been derived from any part of this construction: the present work

constructs only the kinematical and symmetry framework onto which a genuine dynamics --- an action, a Lagrangian, equations of motion --- would need to be built. A more tractable starting point than the full spectral action is a deliberately limited action S = S_lattice + S_Dirac + S_Yukawa, built from only the structures already established in Parts III through VII, tested first for the minimal requirements of consistency --- whether it preserves the Spin(8) symmetry of Part V, whether it admits the chiral structure sought in Part VIII, and whether it breaks to anything resembling the Standard Model gauge group --- before any attempt to reproduce the full Standard Model Lagrangian.

Part IX --- The Search for a Non-Circular Selection Principle

21. Criteria Tested and Their Outcomes

A systematic search for a principle selecting D_4 over E_6 or E_8 without presupposing the answer examined several candidates. Symmetry-efficiency and information-theoretic measures were found underdetermined or undefined. Representation-orbit compression numerically favors D_4 but was found circular, rewarding exactly the property (triality-related multiplicity) already known to distinguish it. Spectral-dimension measures do not discriminate at all, since spectral dimension equals rank for every full-rank lattice examined.

A symmetric-split criterion was tested directly: among the seven ways to split D_8 = so(16) into $D_k \times D_{8-k}$, k = 4 is the unique self-dual point where the two factors are isomorphic. Tested against this document's own Maximum Symmetry Principle, this criterion fails outright: direct computation of $|W(D_k)| \times |W(D_{8-k})|$ shows k = 4 gives the smallest product (36,864) among genuine two-factor splits, less than half that of k = 2 or k = 6 (92,160 each) --- the symmetric split is disfavored, not favored, by the framework's own existing symmetry principle.

22. A Clean, Non-Circular Result

A distinct, representation-theoretic question was tested: for which n does D_n = so(2n) admit dimensional coincidence between its vector representation (dimension 2n) and its half-spin representations (dimension 2^{n-1}) --- the precise, necessary condition for any permutation symmetry to exist among representations of different geometric origin. Direct verification confirms $2n = 2^{n-1}$ has exactly one solution in positive integers: n = 4 ($2\times4 = 8 = 2^3$); for n < 4 the vector representation is larger. That no further coincidence occurs for any n > 4 follows by induction: at n = 5, $2^4 = 16 > 10 = 2\times5$; assuming 2^(k-1) > 2k for some k ≥ 5, it follows that 2^k = 2·2^(k-1) > 4k, and since k ≥ 5 gives 4k > 2k+2 = 2(k+1), we obtain 2^k > 2(k+1), extending the inequality to k+1 and, by induction, to every n > 4.

This gives the clearest non-circular explanation found for why triality can exist only for D_4 among all D_n: triality --- an exact permutation among three representations --- requires those representations to share a common dimension, and this occurs, among all D_n, uniquely at n = 4.

22.1. The Internal D_4 Is Forced, Not Chosen, Once the Spacetime Block Is D_4

A question distinct from the preceding one is worth isolating explicitly: granting that the spacetime block D_{4a} is D_4 (established independently in Parts I–III, by the laminated descent's landmark criteria, with no reference to triality), is the appearance of a second D_4 as the internal block D_4_B --- the block actually carrying triality --- an additional, independent fact requiring its own justification, or a forced consequence of the first?

It is forced. Borel--de Siebenthal theory, already used in Part IV to establish $D_4 \oplus D_4 \subset E_8$, guarantees that any two maximal-rank subalgebras of E_8 of the same isomorphism type are conjugate under the Weyl group $W(E_8)$. Since Weyl group elements act as orthogonal transformations preserving the root system and all inner products, this conjugacy extends to orthogonal complements: if D_{4a} is any embedding of D_4 realized as one factor of a $D_4 \oplus D_4$ maximal subalgebra of E_8, its orthogonal complement within $\Phi(E_8)$ is the image, under the same Weyl conjugation carrying D_{4a} to the standard embedding, of the standard construction's second factor --- and is therefore of D_4 type as well, not merely in the specific coordinate realization used in Part IV but for any such embedding whatsoever. Direct computation confirms this for a second, independently chosen embedding (using coordinates {0,1,4,5} in place of {0,1,2,3}): the resulting subsystem and its orthogonal complement both reproduce D_4's exact root count and inner-product distribution.

The consequence is a genuine reduction in the number of independent facts this framework must postulate. The appearance of a second D_4 hosting triality is not a further coincidence sitting alongside the spacetime block's being D_4; it is a mathematical consequence of the spacetime block's being D_4 at all, via Borel--de Siebenthal alone, with no reference to triality required to derive it. What remains open is exactly one question, not two: why the laminated descent lands on D_4 in the first place (Part II), not a separate question of why the resulting internal structure happens to support triality as well.

23. Necessary and Sufficient Conditions Within the Classification of Simple Lie Algebras

Sections 21--22 show D_4 possesses properties no competitor shares. A materially stronger claim is available: that the properties themselves, demanded in advance, force D_4 uniquely --- not merely among D_n, but among every simple Lie algebra.

Theorem (Necessary and Sufficient Conditions for Triality). A simple Lie algebra $\mathfrak{g}$ admits an outer automorphism of order exactly 3 if and only if $\mathfrak{g}$ is isomorphic to $so(8) = D_4$.

Proof. The outer automorphism group of a simple Lie algebra is isomorphic to the automorphism group of its Dynkin diagram (Humphreys, 1972; Bourbaki, 1968). Checking this across the complete classification: type A_n ($n \geq 2$) has diagram automorphism group of order 2; types B_n, C_n, E_7, E_8, F_4, and G_2 have trivial diagram automorphism group (order 1); type D_n ($n \geq 5$) has order 2; type E_6 has order 2. Every one of these has automorphism group of order at most 2, and a group of order 1 or 2 contains no element of order 3. The single exception is D_4: its diagram --- a central node joined to three outer nodes --- has automorphism group S_3, of order 6, the unique case in the classification containing elements of order 3. No other simple Lie algebra, of any type or rank, admits an order-3 outer automorphism.

This is the converse direction the earlier sections lacked: triality, stated as a property to be demanded independently of any prior preference, admits no solution anywhere in the classification of simple Lie algebras except D_4. Combined with Section 22's explanation of why this holds within the D_n family specifically (the dimension coincidence $2n = 2^{n-1}$ at $n = 4$), this gives a complete, non-circular passage from a stated requirement to a unique structure.

The honest limit of this result, stated precisely. This theorem has the logical form {desired property} implies D_4 --- the converse of {D_4} implies {desired properties} established elsewhere in this document, exactly the direction of implication a selection principle requires. What it does not supply, and what no mathematical theorem can supply on its own, is a derivation of why the desired property (order-3 triality) should be physically demanded in the first place, rather than some other property entirely. That remains a postulate of this framework, named as such in Part X; no amount of further mathematics converts a

postulated physical requirement into a proven one --- only mathematics conditional on that requirement, which is what this theorem provides.

23.1. A Synthesized Structural Statement and Its Epistemic Status

The results established across Parts I through IX can be organized into a single conditional statement: if a candidate structure is required to be (i) a rootless, even, unimodular lattice, (ii) reached by the laminated descent of Part II under its stated branch-selection rule, (iii) a point at which non-abelian diagram symmetry first appears within the root-lattice range, and (iv) capable of supporting an order-three outer automorphism (triality), then the unique structure satisfying all four conditions is D_4. This statement is a synthesis of results proved separately above, not a new theorem in its own right, and two of its four conditions --- (ii)'s branch-selection rule and (iv)'s triality requirement --- remain postulates of this framework rather than derived necessities, exactly as Part X discloses; presenting this synthesis as a single unconditional theorem would overstate what has been shown, and it is not so presented here.

23.2. Future Direction: The Missing Variational Principle

The single most consequential open task identified by this entire investigation is not a new computation but a new kind of result: a variational principle from which the requirement of triality itself --- condition (iv) above --- would follow, rather than being posited. Sections 21 through 23 show that once triality is demanded, D_4 follows uniquely and non-circularly; they do not show why triality should be demanded rather than any other structural property. Locating such a principle, or establishing that none exists within a stated natural class, is identified here as the clearest remaining path toward converting this framework's central postulate into a derived consequence.

An interesting structural parallel appears in the work of Gresnigt, Gourlay, and Varma (2023, European Physical Journal C 83:747), where an order-three automorphism acting within a sedenionic (Cayley--Dickson) framework likewise generates a three-generation structure with nontrivial algebraic relations among the resulting sectors --- a linear dependence among the three sets of triality-related operators comparable in character to the partial overlap documented for this document's own three triality sectors in Part VII. Although the underlying algebraic setting (sedenions and the Clifford algebra Cl(6)) is fundamentally different from the D_4-triality construction developed here, the recurrence of an order-three symmetry producing three interrelated, partially dependent sectors suggests that such structures may arise more generally than within a single mathematical framework. Whether this similarity reflects a deeper shared mathematical principle or is simply a structural analogy between unrelated constructions remains an open question, not resolved here. The same work also constructs its generation structure through a direct linear automorphism action on fermionic ladder operators, rather than through a domain-wall or Landau--Ginzburg mechanism; this suggests an alternative direction worthy of investigation for the triality-covariant Yukawa coupling identified as missing in Part VIII, though whether such a construction transfers to the D_4 lattice setting used here has not been established.

23.3. An Independent Corroboration via Global Representation Theory

A separate line of inquiry, starting from Conway's group Co_0 itself rather than from the laminated descent, offers independent support for the strategy this document follows, without altering any of its results. The question addressed is: does Co_0's defining 24-dimensional representation on the Leech lattice, considered globally, already carry a complex or quaternionic structure from which triality-like phenomena might emerge directly --- or is passage to a smaller substructure genuinely necessary?

Co_0's 24-dimensional representation is documented as absolutely irreducible (irreducible over the complex numbers, not only over the reals). Since Co_0 is defined as a subgroup of the real orthogonal group O(24) preserving the Leech lattice's positive-definite symmetric inner product, this representation carries an invariant real symmetric bilinear form --- the defining signature of real Frobenius--Schur type. Its commutant algebra is therefore $\mathbb{R}$ alone: no complex or quaternionic structure is available at the level of the full Conway group.

This is reported as a consistency check, not a new derivation: it confirms, from an entirely different starting point, that no shortcut exists at the level of Co_0's global representation, and that reaching the complex, triality-supporting structure documented in Parts IV through VII requires descending to a genuine substructure --- exactly the route this document takes, through the laminated sequence and the explicit $E_8 \supset D_4 \oplus D_4$ decomposition, rather than through Co_0's representation theory directly. The two approaches are complementary: one confirms a necessary condition (no global shortcut exists) that the other's explicit construction already satisfies by taking the only route available.

A note on Co_0 versus Co_1 is worth adding for precision. That Co_0 itself, rather than the simple group $Co_1 = Co_0/\{\pm 1\}$, is required for this representation reflects a separate fact --- the central involution acts as $-I$, not as the identity --- and is unrelated to the real-type classification above; the two facts should not be conflated.

24. Relation to Existing Approaches

This pathway is positioned here relative to other research programs, not to argue priority or superiority over them. Superstring theory, the dominant framework for unification, works in ten dimensions and has no established connection to the Leech lattice or D_4; this document diverges from it, as stated at the outset. Loop quantum gravity and causal set theory build spacetime from combinatorial or relational discrete structures without reference to exceptional lattices, sharing this document's general discrete-to-continuum ambition but not its specific starting object. E_8 Theory (Lisi, 2007) and the broader triality-based research program are the closest relatives to Parts IV through VII of this document, and are cited there directly as the source of the speculative bridge to particle physics, not as independent confirmation of it.

A closer and more precise relative to Part V specifically is the trinification family-symmetry program in supersymmetric grand unification, tracing to Glashow's original $[SU(3)]^3 \rtimes \mathbb{Z}_3 \subset E_6$ proposal (1984) and developed in detail by Camargo-Molina, Morais, Ordell, Pasechnik, and Wessén (2018), who construct a supersymmetric model from the identical classical branching $E_8 \supset E_6 \times SU(3)$ used in Part V of this document, explicitly naming the resulting SU(3) a family symmetry SU(3)_F intended to organize three generations. The mechanism is nonetheless distinct from the one developed here: their SU(3)_F is a continuous (global or gauged) symmetry acting on a family index carried by a single, fixed copy of E_6's 27-plet, whereas the triality mechanism of Part VII is discrete ($\mathbb{Z}_3$) and produces three distinct E_6 root systems related cyclically by the verified triality generator, with the spacetime D_4 block held fixed throughout. Both approaches extract a threefold structure from the same $E_8 \supset E_6 \times SU(3)$ branching; whether they are related beyond this shared starting point, or represent genuinely independent mechanisms for the number three, is not established here and is recorded as a further open question this document does not resolve.

Framework	Starting object	Dimension	Triality	Dynamics
Lisi's E8 Theory	E8	248	Yes	Partial (Lagrangian proposed, chirality disputed)
Superstring theory	10D worldsheet

| 10 | No | Yes (established action) | | This framework | Leech lattice | 24 | Yes | Incomplete (kinematics and symmetry only, Part VIII) |

This table is a positioning device, not a ranking; each framework differs in starting assumptions as much as in results, and "incomplete" is stated here about this document's own dynamics with the same directness applied to every other open item in Part X.

Original Contributions of This Work

The individual mathematical objects employed throughout — the Leech lattice, the Golay code, the E_8 root system, D_4 triality, the Gursey–Ramond–Sikivie $E_6/SO(10)$ embedding, the CHSW breaking mechanism — are all classical or previously established elsewhere, as stated in the Introduction. What this document adds, beyond assembling them into one connected pathway, is grouped below by kind, so a reader can distinguish at a glance what is proved, what is computed, what is conceptual framing, and what bears on physical consistency.

Mathematical (proven results)

- the verification, by four independent geometric tests (norm ratio, short-root hexagon angles, long-short angle set, Cartan matrix entry), that this construction's full triality-fixed subalgebra is the exceptional Lie algebra G_2, not merely the SU(3) subsystem of its long roots — identifying the classical origin of the $1/\sqrt{3}$ constant found elsewhere (Part VII, Section 14.1);
- the identification of a strictly conditional, falsifiable-in-principle consequence — six additional color-triplet/antitriplet vector-boson states — following from G_2's standard branching under SU(3), contingent on two explicitly unestablished physical assumptions (Part VII, Section 14.2);
- the mathematical proof that a cyclic group of order 3 admits no fourth orbit, and the general closure-identity proof that no single fixed mass-coupling direction can support a sign change across all three cyclic triality transitions (Part VII–VIII);
- a Necessary-and-Sufficient-Conditions theorem establishing D_4 as the unique simple Lie algebra admitting an order-three outer automorphism, together with the representation-dimension identity $2n = 2^{n-1}$ explaining this uniqueness within the D_n family (Part IX);
- the Half-E_8 Partition theorem: the union of the three triality generations is exactly half of E_8's roots (Part VII, Section 17.2);
- the identification, from published lattice-classification data, that dimension 24 is not merely the first dimension admitting a rootless even unimodular lattice but the last dimension (among those classified, up to 48) where such a lattice is also unique, with the number of rootless competitors at dimension 32 exceeding ten million (Part I, Section 3.1);
- the proof, via Weyl-conjugacy under Borel–de Siebenthal, that the internal D_4 hosting triality is a forced consequence of the spacetime block being D_4, not an independent coincidence (Part IX, Section 22.1);
- a complete proof, by exhaustive case analysis over all 18 non-invariant roots of D_4_B, that consecutive triality-related generators are mutually orthogonal ($\langle \alpha, T\alpha \rangle = 0$) (Part VIII, Section 19.2).

- the identification that the tetrahedral angles of Section 17.3 and the ⅓:⅔ splitting ratio underlying the orthogonality proof of Section 19.2 are a single geometric fact, not two independent results (Part VIII, Section 19.2).

Computational (machine-verified constructions)

- an explicit, machine-verified computation of the $E_8 \supset D_4 \oplus D_4$ root-system decomposition and its associated (8,8)-bimodule structure (Part IV);
- the direct construction and verification of a triality-invariant SU(3) subalgebra, unique among the 96 candidates tested, cross-checked by two independent computational routes (Part VII, Section 14);
- the explicit construction of three cyclically related $E_6 \times SU(3)$ breakings, and the complete resolution of their overlap structure into a symmetric 24+16+16+16 decomposition, together with a representation-theoretic derivation of the value 16 from SU(3) branching (Part VII, Sections 16–17);
- the identification and computational refutation of the symmetric-split criterion (Part IX, Section 21) and of an external claim that the three-generation structure follows automatically from $\mathbb{Z}_3$ symmetry alone (Part VIII, Section 19.1);
- the discovery that the three distinct types of 16-root sector within the triality partition share an identical internal inner-product signature, matching the structural pattern of a minuscule spinor representation (Part VII, Section 17.1);
- the discovery that the triality-invariant SU(3) plane relates to all 96 candidate SU(3) planes through exactly two nonzero principal angles, both exact tetrahedral constants, with the full basis-change matrix verified orthogonal with exact algebraic entries (Part VII, Section 17.3).

Physical (consistency and construction)

- an exact, closed-form continuum solution of the domain-wall chiral zero mode, exponentially localized and of exact algebraic chirality, realized directly with this document's own Cl(4) Dirac matrices (Part VIII, Section 19.1);
- the verification that D_4/Spin(8)'s three fundamental representations are of real Frobenius–Schur type — the same feature underlying the Distler–Garibaldi obstruction against E_8 — together with the demonstration that this construction's specific use of triality avoids that obstruction (Part V, Section 11.1);
- the self-referential wall construction achieving an exact, equal-magnitude sign change across all three triality transitions using no coupling beyond the construction's own generators (Part VIII, Section 19.2).

Conceptual (framework and methodology)

- the Maximum Symmetry Selection Hypothesis, named and explicitly distinguished from a proof, governing branch selection in the laminated descent (Part II).

This grouping is provided to let a reader distinguish, at a glance, this document's own contribution from the established mathematics it builds upon.

Part X --- Status

25. What Is Established, What Is Postulated, What Remains Open

Established:

- The Golay code and Leech lattice's defining properties and Leech's uniqueness under the stated criteria.
- The physical reading of dimension 24 via modular forms and the Schellekens VOA classification (with the residual FLM conjecture named).
- The loss of lattice uniqueness beyond dimension 24 (over ten million rootless competitors at dimension 32, King), confirming 24 as the last, not merely the first, dimension where rootlessness selects a single lattice.
- The laminated sequence and its landmark identifications.
- Pointed Gromov–Hausdorff convergence of D_4 to Euclidean $\mathbb{R}^4$; the failure of graph distance and success of the diffusive/covariance metric, confirmed computationally.
- $E_8 \supset D_4 \oplus D_4$, confirmed by direct root-system computation; the $E_8 \to E_6 \times SU(3)$ breaking, verified computationally and anchored in CHSW (1985).
- The full explicit chain to one Standard Model generation's exact quantum numbers via $E_7 \to E_6 \to 27 \to SO(10)$, independently reproduced on a second octad.
- Independent mathematical consistency with Harvey–Moore's heterotic construction, sharpened to an explicit, verified Weyl(E_8) isometry.
- The SU(3) subalgebra of the internal D_4 (unique among 96 candidates tested) invariant under a correctly-constructed and verified triality generator; the Nielsen–Ninomiya and Distler–Garibaldi obstructions.
- The new three-fold cyclic triality-sector structure of Part VII (a possible, not established, interpretation as three fermion generations), with its overlap pattern fully resolved (24 + 16 + 16 + 16, verified) and the further discovery that all three 16-root sector types share an identical internal inner-product signature matching a minuscule-representation pattern.
- The direct test, and rejection, of an exact isomorphism between these 16-root sectors and the explicit SO(10) spinor of Part V, whose internal signature was computed and found genuinely different.
- The discovery that the triality-invariant SU(3) plane relates to all 96 candidate planes via exactly two nonzero principal angles, both exact tetrahedral constants, with an exact-algebraic-entry basis-change matrix verified between invariant and generation frames.
- The identification, confirmed by four independent geometric tests, that the full triality-fixed subalgebra is G_2 rather than SU(3) alone, explaining the classical origin of this constant.
- The further finding that the three generations' union covers exactly half of E_8's roots, with D_4_B entirely excluded.
- The verification that D_4/Spin(8)'s representations share E_8's real-type chirality obstruction, and that this construction's specific use of triality avoids it.
- The proof that the internal D_4 hosting triality is forced by Weyl-conjugacy once the spacetime block is D_4, reducing two apparently independent facts to one.
- The dimension-matching identity $2n = 2^{n-1}$, and the Necessary-and-Sufficient-Conditions theorem establishing D_4 as the unique simple Lie algebra admitting an order-3 outer automorphism $\Longrightarrow n = 4$.

- An exact, closed-form continuum solution of the domain-wall chiral zero mode using real Cl(4) matrices, exponentially localized and of exact chirality by algebraic construction; a genuine kink solution of a $\mathbb{Z}_3$-symmetric Landau–Ginzburg potential producing a non-arbitrary, sign-changing fermion mass function.
- The complete, exhaustively-verified proof that all 18 non-invariant D_4_B roots satisfy consecutive-generator orthogonality, yielding a self-referential wall construction with exact, equal-magnitude sign changes across all three triality transitions and no external coupling.
- The discovery that a single fixed mass-coupling direction supports a sign change across only two of the three cyclic triality transitions, not three, refuting the strongest version of an automatic three-generations claim.

Postulated: That even, unimodular, rootless lattices are the physically relevant class; that the largest-automorphism-group branch is selected where the laminated construction is non-unique; that the physically relevant symmetry structure is the continuous group Spin(8) rather than the finite lattice automorphism group alone; that triality is required of the physically relevant structure in the first place.

Open:

- Why the laminated descent lands on D_4 specifically, rather than E_6 or E_8, as the physically privileged endpoint (Section 22.1 reduces this to a single residual question by showing the internal D_4 follows automatically once this one is answered) — the symmetric-split criterion was tested and found to conflict with this document's own symmetry principle, and no candidate tested resolves this non-circularly.
- Whether the three cyclic triality sectors of Part VII correspond to physical fermion generations, and what physical role, if any, their pairwise-shared sectors play.
- A lattice discretization of the exact continuum zero mode of Part VIII compatible with the Nielsen–Ninomiya constraint — a standard challenge in lattice field theory generally, not undertaken to completion here.
- An explicit spatial lattice path realizing the self-referential mass construction of Section 19.2, now built on a fully proven orthogonality fact for D_4_B, and an extension of that proof to other triality generators or root systems, not attempted here.
- A spectral-triple construction realizing dynamics on this construction's own data.
- Any determination of physical coupling constants or particle masses.
- Whether SU(3)_color embeds as the specific maximal SU(3) subgroup of G_2 identified in Section 14.2, and whether G_2 itself is physically realized as a gauge symmetry — the two assumptions on which the conditional six-state prediction of that section depends.
- Any prediction distinguishing this construction, observationally, from an alternative — though one candidate is already implicit in Part VII: if the cyclic triality-sector mechanism is validated as the source of fermion generations, it forbids a fourth generation outright, since a cyclic group of order 3 admits no fourth orbit member (Part VII, Section 16); this is named as the single clearest candidate for a falsifiable consequence, contingent on resolving Part VIII's open items first, not as an established prediction.

This document is offered as a rigorously constructed, honestly bounded mathematical and computational exploration --- not as a proof that the observed universe is a projection from a twenty-four-dimensional foundation. That claim, should it ever be established, will require experimental evidence no mathematical

construction, however precise, can supply on its own. The principal contribution of this manuscript is not the discovery of new algebraic objects, but the construction of a logically connected and computationally reproducible framework linking well-established mathematical structures into a single speculative pathway from the Leech lattice to four-dimensional triality-based particle symmetry. Every theorem, computation, hypothesis, and open problem has been explicitly distinguished to facilitate independent verification.

Appendix --- Computational Reproducibility

Every computational claim in this document (E_8's 240-root system and its $D_4 \oplus D_4$ decomposition, the $E_6 \times SU(3)$ breaking, the order-three triality generator and its action on all 96 candidate SU(3) subalgebras, the three-fold cyclic triality-sector structure and its 24+16+16+16 overlap decomposition, and the Weyl(D_4) symmetry checks of Part III) was produced by direct enumeration in Python using standard numerical linear algebra (NumPy), operating on explicit coordinate representations of E_8 and D_4 root systems. No specialized computer-algebra system or unpublished code library was used. The computations are individually short (root generation by explicit coordinate formulas, inner products, orbit enumeration under a constructed linear map) and are described in the text in sufficient detail --- generator matrices, coordinate conventions, and selection rules --- for independent reimplementation and verification from those descriptions alone. Throughout this document, the term verified refers to this author's own direct computation, reported with sufficient detail for independent reimplementation, rather than to independent third-party verification, which has not been sought or obtained.

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